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FINITE-ROTATION ELEMENTS FOR THE NON-LINEAR ANALYSIS OF THIN SHELL STRUCTURES

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Abstract—For the numerical analysis of shells undergoing finite rotations doubly curved finite shell elements are developed via the displacement formulation. The derivation starts from a consistent finite-rotation shell theory which is transformed by a variational procedure into an incremental formulation. Thus, the non-linearity can be treated by an incremental-iterative technique. The non-linear element matrices are obtained by a tensor-oriented procedure permitting a direct transformation of the initial equations into efficient numerical models. Unlike in the usual procedure, the KIRCHHOFF–LOVE assumption is treated as a subsidiary condition at the element level. This computer-oriented approach permits the elimination of the dependent rotational degrees of freedom without loss of accuracy. Finally, some examples are presented to demonstrate the ability of the resulting finite elements to deal with finite-rotation problems.

1. INTRODUCTION

Thin shell structures which are used increasingly in various branches of modern technology may undergo finite deformations, in particular finite rotations. The numerical analysis of these phenomena is therefore of significant practical importance.

In recent years a number of finite elements have been developed for the non-linear analysis of shell structures (see e.g. Bathe and Bolourchi, 1980; Surana, 1983; Hughes and Liu, 1981; Oliver and Onate, 1984), some of them on the basis of tensorial shell theories (see e.g. Harte, 1982 and Nolte, 1983). In the analysis of strongly non-linear problems they may, however, lead to significantly different numerical results as has been demonstrated by the systematic numerical comparisons presented by Nolte (1983). At present, it is not possible to decide conclusively which numerical models or theories will lead to the most reliable results if finite rotations are involved. For this reason the development of further refined finite elements in connection with systematic numerical studies still seems to be needed. For the development of finite elements applicable to general finite-rotation problems, which will be referred to *finite-rotation elements*, special care should be taken both in the formulation of the theory and in the selection of the mathematical approach, because only an optimal combination of both can lead to satisfactory high-precision finite elements.

The non-linear shell theories presented in the literature vary significantly (see e.g. Pietraszkiewicz, 1984; Koiter, 1966; Shapovalov, 1968; Stein *et al.*, 1982; Başar and Krätzig, 1985). In particular, many of them are simplified theories which are not suitable for the derivation of finite-rotation elements. In the present derivation we shall use the theory presented by Başar (1987), adopting the KIRCHHOF–LOVE hypothesis (see Başar and Krätzig, 1989; Ding, 1989). In this theory, no simplifications have been made based on assumptions concerning the order of magnitude of the deformation variables. Thus, it should be applicable to a wide range of problems. Furthermore, it satisfies the requirements imposed on consistent formulations and gives a clear physical interpretation of the force variables. The objective of this paper is the transformation of this theory into a finite element formulation and to demonstrate its numerical efficiency.

In the derivation of finite elements a tensor-oriented procedure has been adopted which has also been used successfully for the development of the NACS-family of elements by Harte and Eckstein (1986). According to this concept the incremental form of the equations can be directly transformed into efficient numerical models. This concept also permits the use of arbitrary displacement approximations and thus the development of an entire element family in a unified consistent manner.

Non-linear terms will be treated by an incremental-iterative procedure, using unbalanced forces which are calculated from the exact non-linear equations. When these forces vanish, the non-linear shell equations will be satisfied automatically.

In the development of non-linear finite elements using the KIRCHHOFF–LOVE hypothesis, the essential problem is the elimination of the rotation vector (the difference vector) without loss of accuracy. In the present formulation, the non-linear KIRCHHOFF–LOVE condition will not be expanded into a TAYLOR-series as has been done, e.g. by Harte (1982) to ensure an *a priori* elimination of the rotational variables. In the case of finite rotations this procedure is inconvenient and introduces errors in the unbalanced forces appearing in the iteration procedure. Instead, a remarkable accuracy can be achieved if the conditions in question are considered at the element level. To do this, the KIRCHHOFF–LOVE hypothesis has been expressed by two sets of equivalent conditions. One of them is used in the form of linear variational equations for the elimination of the incremental rotational variables. The second non-linear one is needed for the exact calculation of the rotation vector of the fundamental state. Thus, the rotation vector can be evaluated to the same degree of accuracy as the displacements and the other kinematic quantities. This computer-oriented concept ensures very satisfactory results even up to rotational angles of 620° and is not much more time consuming than the usual *a priori* elimination of the rotational degrees of freedom.

The use of a shell theory instead of the isoparametric concept presents many significant advantages for the development of efficient numerical models. The consistency of a shell theory as a two-dimensional approximation can be checked from the theoretical point of view. In the classical concept, the variables used in the numerical implementation are those defined on the shell element. Thus, their later transformation into geometrically interpretable ones is superfluous. In particular, shell theories permit the calculation of the real force variables acting on the middle surface. For engineering applications, this aspect is very significant, especially when dealing with finite-rotation problems, where the definition of geometrically interpretable forces requires special attention (see, e.g. Başar, 1987). Tensorial shell theories permit, finally, an exact description of the shell geometry. The calculation of the corresponding geometrical variables is not necessarily much more time consuming than the approximate description of the shell geometry used in the isoparametric concept. As has been pointed out in Harte and Eckstein (1986), such an approximation may cause shape deviations and, thus, significant errors in the analysis of imperfection-sensitive structures.

2. BASIC EQUATIONS OF A FINITE-ROTATION SHELL THEORY

In this section, the basic equations of the geometrically non-linear theory given by Başar (1987) are presented under the KIRCHHOFF–LOVE hypothesis. Points of the undeformed middle surface $\mathring{F}$ are described by the position vector $\mathring{\mathbf{r}} = \mathring{\mathbf{r}}(\theta^\alpha)$ with convective curvilinear coordinates θ^α. Similar to $\mathring{\mathbf{r}}$, all the geometrical elements associated with the undeformed state $\mathring{F}$ will be denoted by the suffix ($^\circ$). Thus (see Başar and Krätzig, 1985):

base vectors: $\mathring{\mathbf{a}}_\alpha$, $\mathring{\mathbf{a}}^\alpha$,	unit normal vector: $\mathring{\mathbf{a}}_3$,
metric tensor: $\mathring{a}_{\alpha\beta}$, $\mathring{a}^{\alpha\beta}$,	determinant of $\mathring{a}_{\alpha\beta}$: $\mathring{a}$,
curvature tensor: $\mathring{b}_{\alpha\beta}$, $\mathring{b}^\beta_\alpha$,	CHRISTOFFEL symbols: $\mathring{\Gamma}^\lambda_{\alpha\beta}$.

Suffix-free symbols refer to the deformed state F. Covariant derivatives with respect to the undeformed middle surface $\mathring{F}$ will be denoted by $(\ldots)|_\alpha$, partial derivatives with respect to θ^α by $(\ldots)_{,\alpha}$. We further introduce along the boundary curve $\mathring{C}$ of the undeformed middle surface the unit tangent vector $\mathring{\mathbf{t}}$ and the unit outward normal $\mathring{\mathbf{u}}$

$$\mathring{\mathbf{u}} = \mathring{\mathbf{t}} \times \mathring{\mathbf{a}}_3 = \mathring{u}_\alpha \mathring{\mathbf{a}}^\alpha, \quad \mathring{\mathbf{t}} = \frac{d\mathring{\mathbf{r}}}{d\mathring{s}} = \mathring{t}_\alpha \mathring{\mathbf{a}}^\alpha, \tag{1}$$

which form together with $\mathring{\mathbf{a}}_3$ a right-handed vector triad $(\mathring{\mathbf{u}}, \mathring{\mathbf{t}}, \mathring{\mathbf{a}})$ and which will be used for the definition of the physical boundary variables.

Under the KIRCHHOFF–LOVE hypothesis the state of deformation of the shell is described by the first and second strain tensors $\alpha_{\alpha\beta}$ and $\omega_{\alpha\beta}$, respectively. These variables are related to the displacement vector $\mathbf{v}$ and the difference vector $\mathbf{w}$

$$\mathbf{v} = \mathbf{r} - \mathring{\mathbf{r}} = v_\alpha \mathring{\mathbf{a}}^\alpha + v_3 \mathring{\mathbf{a}}^3, \quad \mathbf{w} = \mathbf{a}_3 - \mathring{\mathbf{a}}_3 = w_\alpha \mathring{\mathbf{a}}^\alpha + w_3 \mathring{\mathbf{a}}^3 \tag{2}$$

by the kinematic relations:

$$\begin{aligned}
\alpha_{\alpha\beta} &= \tfrac{1}{2}(\varphi_{\alpha\beta} + \varphi_{\beta\alpha} + \varphi_{\alpha\lambda}\varphi_{\beta\cdot}^{\cdot\lambda} + \varphi_{\alpha 3}\varphi_{\beta 3}),\\
\omega_{\alpha\beta} &= \tfrac{1}{2}[w_\alpha|_\beta + w_\beta|_\alpha - \mathring{b}_\alpha^\lambda \varphi_{\beta\lambda} - \mathring{b}_\beta^\lambda \varphi_{\alpha\lambda} - 2\mathring{b}_{\alpha\beta} w_3 + \varphi_{\beta\cdot}^{\cdot\lambda}(w_\lambda|_\alpha - \mathring{b}_{\lambda\alpha} w_3)\\
&\quad + \varphi_{\alpha\cdot}^{\cdot\lambda}(w_\lambda|_\beta - \mathring{b}_{\lambda\beta} w_3) + \varphi_{\beta 3}(w_{3,\alpha} + \mathring{b}_\alpha^\lambda w_\lambda) + \varphi_{\alpha 3}(w_{3,\beta} + \mathring{b}_\beta^\lambda w_\lambda)]\\
&= -[\varphi_{\alpha 3}|_\beta + \mathring{b}_\beta^\rho \varphi_{\alpha\rho} + w_3(\mathring{b}_{\alpha\beta} + \varphi_{\alpha 3}|_\beta + \mathring{b}_\beta^\rho \varphi_{\alpha\rho}) + w_\rho(\varphi_{\alpha\cdot}^{\cdot\rho}|_\beta - \mathring{b}_\beta^\rho \varphi_{\alpha 3})],
\end{aligned} \tag{3}$$

where the abbreviations

$$\varphi_{\alpha\rho} = v_\rho|_\alpha - \mathring{b}_{\rho\alpha} v_3, \quad \varphi_{\alpha 3} = v_{3,\alpha} + \mathring{b}_\alpha^\lambda v_\lambda, \quad \varphi_{\alpha\cdot}^{\cdot\beta} = \varphi_{\alpha\rho}\mathring{a}^{\rho\beta} \tag{4}$$

are the so-called deformation gradients. By means of the KIRCHHOFF–LOVE hypothesis $[\mathbf{a}_\alpha \cdot \mathbf{a}_3 = (\mathbf{a}_\alpha \cdot \mathbf{a}_3)|_\alpha = 0]$, both expressions given above for $\omega_{\alpha\beta}$ are equivalent. We shall, however, use in the following the second one which does not contain the derivatives of the difference vector $\mathbf{w}$.

Again due to the KIRCHHOFF–LOVE hypothesis, the difference vector w_i and accordingly the rotation vector ω_α defined by the transformations

$$\omega_\alpha = \mathring{\varepsilon}_{\beta\alpha} w^\beta, \quad w_\alpha = \mathring{\varepsilon}_{\alpha\beta}\omega^\beta \tag{5}$$

may be expressed in terms of the displacements v_i. The constraints valid for w_i are of the form (see, e.g., Başar and Krätzig, 1989; Ding, 1989):

$$w_\beta = -\varepsilon^{-1/2}(\varphi_{\beta 3} + \varphi_{\alpha\cdot}^{\cdot\alpha}\varphi_{\beta 3} - \varphi_{\beta\cdot}^{\cdot\alpha}\varphi_{\alpha 3}), \quad w_3 = \varepsilon^{-1/2}(1 + \varphi_{\rho\cdot}^{\cdot\rho} + \det \varphi_{\alpha\cdot}^{\cdot\beta}) - 1 \tag{6}$$

with

$$\varepsilon = \frac{a}{\mathring{a}} = 1 + 2(\alpha_\lambda^\lambda + \delta_{\lambda\mu}^{\alpha\beta}\alpha_\alpha^\lambda \alpha_\beta^\mu), \quad \alpha_\beta^\alpha = \alpha_{\lambda\beta}\mathring{a}^{\alpha\lambda}. \tag{7}$$

They can, however, be replaced by the conditions

$$\mathbf{a}_\alpha \cdot \mathbf{a}_3 \equiv w_\rho(\delta_\alpha^\rho + \varphi_{\alpha\cdot}^{\cdot\rho}) + \varphi_{\alpha 3}(1 + w_3) = 0, \quad \mathbf{a}_3 \cdot \mathbf{a}_3 - 1 = w_3(2 + w_3) + w_\alpha w^\alpha = 0 \tag{8}$$

which we shall transform into linear variational eqns (19) and (20) in order to eliminate the incremental components $\delta w_i = \overset{+}{w}_i$ and $\delta^2 w_i = \overset{++}{w}_i$ at the element level. In contrary, the first form (6) will be used for the calculation of the difference vector of the fundamental state w_i. Thus, the simultaneous use of the both sets of conditions (6) and (19), (20) enables a very accurate elimination of the difference vector and thus avoids the usual approach of the difference vector by TAYLOR series (see, e.g., Harte and Eckstein, 1986; Başar and Krätzig, 1985).

For the derivation of the tangent stiffness relation which expresses the equilibrium of a finite displacement model we employ the principle of virtual work, being in the case of finite rotations of the form

$$\delta^* A = \int\int_{\mathring{F}} (p^\alpha \delta v_\alpha + p^3 \delta v_3)\, d\mathring{F} + \oint_{\mathring{C}} (n^\alpha \delta v_\alpha + n^3 \delta v_3 + m^\alpha \delta w_\alpha)\, d\mathring{s} - \int\int_{\mathring{F}} (\tilde{N}^{(\alpha\beta)} \delta \alpha_{\alpha\beta} + M^{(\alpha\beta)} \delta \omega_{\alpha\beta})\, d\mathring{F} = 0, \quad (9)$$

where

$\tilde{N}^{(\alpha\beta)}$, $M^{(\alpha\beta)}$ = pseudo stress resultant tensor, moment tensor,
n^i, m^α = tensorial boundary forces, moments,
p^i = load components defined with respect of $\mathring{\mathbf{a}}_i$,
$d\mathring{F}$ = element of the undeformed middle surface $\mathring{F}$,
$d\mathring{s}$ = element of the undeformed boundary curve $\mathring{C}$.

The second integral in (9) which expresses the virtual work of conservative boundary loads may be given alternatively as

$$\oint_{\mathring{C}} (n^\alpha \delta v_\alpha + n^3 \delta v_3 + m^\alpha \delta w_\alpha)\, d\mathring{s} = \oint_{\mathring{C}} \left[n_t \delta v_t + n_u \delta v_u + n_3 \delta v_3 + \frac{1}{1+w_3} (m_t \delta \omega_t + m_u \delta \omega_u) \right] \quad (10)$$

in terms of physical boundary variables $(n_t, n_u, \ldots)$ defined with respect to the orthogonal vector triad $\mathring{\mathbf{u}}$, $\mathring{\mathbf{t}}$ and $\mathring{\mathbf{a}}_3$ (1). In view of the KIRCHHOFF–LOVE constraints (6), (8), the component $\delta\omega_u$ cannot be prescribed independently along $\mathring{C}$. We note that the factor $1/1+w_3 = 1/\cos\omega$ with the angle of rotation ω is approximated in the moderate-rotation theories (Harte, 1982; Başar and Krätzig, 1985) by unity. Its consideration however is of significant importance for an accurate analysis of finite-rotation phenomena induced by load couples.

We have finally to formulate the constitutive equations needed for the elimination of the internal forces $\tilde{N}^{(\alpha\beta)}$ and $M^{(\alpha\beta)}$ in (9). If we assume an isotropic HOOKEAN material and sufficiently small strains, they are given by

$$\tilde{N}^{(\alpha\beta)} = DH^{\alpha\beta\rho\lambda} \alpha_{\rho\lambda}, \quad M^{(\alpha\beta)} = BH^{\alpha\beta\rho\lambda} \omega_{\rho\lambda}, \quad (11)$$

$$H^{\alpha\beta\rho\lambda} = \frac{1-\nu}{2} \left(\mathring{a}^{\alpha\lambda} \mathring{a}^{\beta\rho} + \mathring{a}^{\alpha\rho} \mathring{a}^{\beta\lambda} + \frac{2\nu}{1-\nu} \mathring{a}^{\alpha\beta} \mathring{a}^{\rho\lambda} \right), \quad (12)$$

$$D = \frac{Eh}{1-\nu^2}, \quad B = \frac{Eh^3}{12(1-\nu^2)}, \quad (13)$$

where h is the thickness of the shell, E is YOUNG's modulus and ν is POISSON's ratio.

3. INCREMENTAL FORMULATION

For an incremental-iterative treatment of the non-linear problem, the equations presented above, in particular the variational principle (9), have to be transformed into incremental relations by a variational procedure. The derivation can be performed systematically if we introduce the following three sets of deformation (Fig. 1): The *initial state* (IS) is the unloaded, undeformed and stress-free reference configuration of the shell. The *fundamental state* (FS) is the equilibrium state of the shell under given loads. This state is supposed to be described by a displacement field v_i satisfying the non-linear principle (9) approximately. The *adjacent state* (AS) is a state defined by a displacement field $\bar{v}_i$ of the

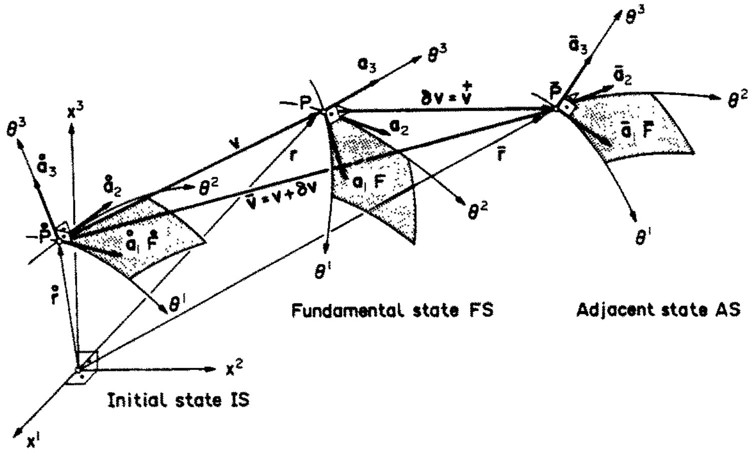


Fig. 1. States of deformations.

form

$$\bar{v}_i = v_i + \delta v_i = v_i + \overset{+}{v}_i \tag{14}$$

where $\delta v_i = \overset{+}{v}_i$ is a special variation. Accordingly, the mechanical variables of this state which are non-linear functions of $\bar{v}_i$ may be replaced by infinite power series in terms of the first- and the higher-order variations, for instance in the form:

$$\bar{\omega}_{\alpha\beta} = \omega_{\alpha\beta} + \delta\omega_{\alpha\beta} + \tfrac{1}{2}\delta^2\omega_{\alpha\beta} + \cdots = \omega_{\alpha\beta} + \overset{+}{\omega}_{\alpha\beta} + \tfrac{1}{2}\overset{++}{\omega}_{\alpha\beta} + \cdots, \tag{15}$$

while the deformation gradients $\bar{\varphi}_{\alpha\beta}$ and $\bar{\varphi}_{\alpha 3}$ are given according to (4) by expressions of the form (14).

Now, we specify the principle (9) for the AS in order to express all the variables of the AS according to (14) and (15). If we neglect the terms of third and higher order with respect to the variations $\overset{+}{v}_i$ and assume that the loads are conservative ($\overset{+}{p}^i = \overset{+}{n}_t = \cdots = \overset{+}{m}_u = 0$), we finally obtain

$$\begin{aligned}
&\iint_{\mathring{F}} (\overset{+}{\tilde{N}}{}^{(\alpha\beta)}\delta\overset{+}{\alpha}_{\alpha\beta} + \overset{+}{M}{}^{(\alpha\beta)}\delta\overset{+}{\omega}_{\alpha\beta})\, d\mathring{F} \rightarrow \mathbf{k}_e \\
&\quad + \frac{1}{2}\iint_{\mathring{F}} (\tilde{N}^{(\alpha\beta)}\delta\overset{++}{\alpha}_{\alpha\beta} + M^{(\alpha\beta)}\delta\overset{++}{\omega}_{\alpha\beta})\, d\mathring{F} + \oint_{\mathring{C}} \left[\frac{\overset{+}{w}_3}{(1+w_3)^2}(m_t\delta\overset{+}{\omega}_t + m_u\delta\overset{+}{\omega}_u)\right. \\
&\quad \left. - \frac{1}{2(1+w_3)}(m_t\delta\overset{++}{\omega}_t + m_u\delta\overset{++}{\omega}_u)\right] d\mathring{s} \rightarrow \mathbf{k}_g \\
&= \iint_{\mathring{F}} (p^\alpha\delta\overset{+}{v}_\alpha + p^3\delta\overset{+}{v}_3)\, d\mathring{F} + \int_{\mathring{C}} \left[n_t\delta\overset{+}{v}_t + n_u\delta\overset{+}{v}_u + n_3\delta\overset{+}{v}_3\right. \\
&\quad \left. + \frac{1}{1+w_3}(m_t\delta\overset{+}{\omega}_t + m_u\delta\overset{+}{\omega}_u)\right] d\mathring{s} \rightarrow \mathbf{p}_e \\
&\quad - \iint_{\mathring{F}} (\tilde{N}^{(\alpha\beta)}\delta\overset{+}{\alpha}_{\alpha\beta} + M^{(\alpha\beta)}\delta\overset{+}{\omega}_{\alpha\beta})\, d\mathring{F} \rightarrow \mathbf{p}_i
\end{aligned} \tag{16}$$

where δ implies a variation with respect to $\overset{+}{v}_i$. This equation contains all submatrices of the tangent stiffness relation, namely the elastic matrix $\mathbf{k}_e$, the geometric matrix $\mathbf{k}_g$, the vector of external nodal forces $\mathbf{p}_e$ and the vector of internal nodal forces $\mathbf{p}_i$. The last two variables $\mathbf{p}_e$ and $\mathbf{p}_i$ occurring at the right-hand side are the so-called *unbalanced* forces to be iterated to zero in the iterative procedure.

The constraints to be satisfied by the incremental variables appearing in (16) can be derived by a unified variational procedure. From (3), we find for instance

$$\overset{+}{\omega}_{\alpha\beta} = -[(1+w_3)(\overset{+}{\varphi}_{\alpha3}|_\beta+\overset{\circ}{b}{}^\rho_\beta\overset{+}{\varphi}_{\alpha\rho})+w_\rho(\overset{+}{\varphi}_\alpha{}^\rho|_\beta-\overset{\circ}{b}{}^\rho_\beta\overset{+}{\varphi}_{\alpha3}) + (\overset{\circ}{b}_{\alpha\beta}+\varphi_{\alpha3}|_\beta+\overset{\circ}{b}{}^\rho_\beta\varphi_{\alpha\rho})\overset{+}{w}_3+(\varphi_\alpha{}^\rho|_\beta-\overset{\circ}{b}{}^\rho_\beta\varphi_{\alpha3})\overset{+}{w}_\rho] \quad (17)$$

and

$$\overset{++}{\omega}_{\alpha\beta} = -2\left[\overset{+}{w}_3(\overset{+}{\varphi}_{\alpha3}|_\beta+\overset{\circ}{b}{}^\rho_\beta\overset{+}{\varphi}_{\alpha\rho})+\overset{+}{w}_\rho(\overset{+}{\varphi}_\alpha{}^\rho|_\beta-\overset{\circ}{b}{}^\rho_\beta\overset{+}{\varphi}_{\alpha3}) + (\overset{\circ}{b}_{\alpha\beta}+\varphi_{\alpha3}|_\beta+\overset{\circ}{b}{}^\rho_\beta\varphi_{\alpha\rho})\frac{\overset{++}{w}_3}{2}+(\varphi_\alpha{}^\rho|_\beta-\overset{\circ}{b}{}^\rho_\beta\varphi_{\alpha3})\frac{\overset{++}{w}_\rho}{2}\right]. \quad (18)$$

The variations of the difference vector $\overset{+}{w}_i$ and $\overset{++}{w}_i$ occurring in this relation are, in view of (8), subjected to

$$\overset{+}{w}_\rho l_\alpha{}^\rho = -[\overset{+}{\varphi}_\alpha{}^\rho w_\rho+\overset{+}{\varphi}_{\alpha3}(1+w_3)],$$
$$\overset{+}{w}_3(1+w_3) = -\overset{+}{w}_\alpha w^\alpha \quad (19)$$

and

$$\overset{++}{w}_\rho l_\alpha{}^\rho = -(\overset{+}{w}_\rho \overset{+}{l}_\alpha{}^\rho+\overset{+}{\varphi}_\alpha{}^\rho\overset{+}{w}_\rho+\overset{+}{w}_3\overset{+}{\varphi}_{\alpha3}),$$
$$\overset{++}{w}_3(1+w_3) = -(\overset{+}{w}_3\overset{+}{w}_3+\overset{+}{w}_\alpha\overset{+}{w}{}^\alpha+\overset{++}{w}_\alpha w^\alpha) \quad (20)$$

where

$$l_\alpha{}^\rho = \delta^\rho_\alpha+\varphi_\alpha{}^\rho-\frac{w^\rho}{1+w_3}\varphi_{\alpha3}. \quad (21)$$

These equations, which are linear in the variational quantities $\overset{+}{w}_i$ and $\overset{++}{w}_i$, will be used in the finite element implementation for the pointwise calculation of the shape functions of $\overset{+}{w}_i$ and $\overset{++}{w}_i$. For evaluation of the fundamental state variables w_i and $l_\alpha{}^\rho$ entering in these relations, we shall employ the exact non-linear equations (6), (7) and (21) which can therefore be satisfied by a suitable iteration in every desired order of accuracy.

Finally, we give the incremental constitutive equations which are in accordance with (11) of the form:

$$\overset{+}{N}{}^{(\alpha\beta)} = DH^{\alpha\beta\rho\lambda}\overset{+}{\alpha}_{\rho\lambda}, \quad \overset{+}{M}{}^{(\alpha\beta)} = BH^{\alpha\beta\rho\lambda}\overset{+}{\omega}_{\rho\lambda}. \quad (22)$$

All further incremental relations can be established by a similar procedure and will not be presented here.

4. DEVELOPMENT OF ELEMENT MATRICES IN TENSOR FORMULATION

In order to develop the element matrices from the incremental principle (16) we employ the tensor-oriented procedure which has been used for the derivation of the NACS-family of elements by Harte and Eckstein (1986). Accordingly, the following relations are presented in index form rather than in the more classical matrix notation. This has the advantage that the formulation of the shape functions and the element matrices is much more transparent and exhibits the mechanical significance of these variables quite clearly.

According to the finite element displacement method we approximate the unknown displacement $\overset{+}{v}_i$ by shape functions v_i^N and associated nodal displacements $\overset{+}{v}^N$. This can be expressed in the form:

$$\overset{+}{v}_\alpha = \sum_{N=1}^{NN} v_\alpha^N \overset{+}{v}^N, \quad \overset{+}{v}_3 = \sum_{N=1}^{NN} v_3^N \overset{+}{v}^N. \tag{23}$$

Here, the upper index N refers to the finite element formulation while NN gives the number of degrees of freedom. The usual partial differentiation of v_α^N and v_3^N leads to the shape functions of the partial derivatives $\overset{+}{v}_{\alpha,\beta}$ and $\overset{+}{v}_{3,\beta}$:

$$\overset{+}{v}_{\alpha,\beta} = \sum_{N=1}^{NN} (v_\alpha^N)_{,\beta} \overset{+}{v}^N = \sum_{N=1}^{NN} v_{\alpha,\beta}^N \overset{+}{v}^N,$$

$$\overset{+}{v}_{3,\beta} = \sum_{N=1}^{NN} (v_3^N)_{,\beta} \overset{+}{v}^N = \sum_{N=1}^{NN} v_{3,\beta}^N \overset{+}{v}^N. \tag{24}$$

From now on, the derivation of all further shape functions can be carried out via tensor calculus by applying the tensorial equations established above. Inserting, for instance, (23) and (24) into the well-known relationship

$$\overset{+}{v}_\alpha|_\beta = \sum_{N=1}^{NN} (v_{\alpha,\beta}^N - \overset{\circ}{\Gamma}{}_{\alpha\beta}^{\lambda} v_\lambda^N)\overset{+}{v}^N = \sum_{N=1}^{NN} v_\alpha^N|_\beta \overset{+}{v}^N, \tag{25}$$

we obtain the shape functions of the covariant derivatives $v_\alpha^N|_\beta$. For the derivation of the shape functions of the deformation gradients we first transform relations (4) into incremental equations. Using equations (23), (24) and (25), this gives:

$$\overset{+}{\varphi}_{\alpha\rho} = \sum_{N=1}^{NN} (v_\rho^N|_\alpha - \overset{\circ}{b}_{\rho\alpha} v_3^N)\overset{+}{v}^N = \sum_{N=1}^{NN} \varphi_{\alpha\rho}^N \overset{+}{v}^N,$$

$$\overset{+}{\varphi}_{\alpha 3} = \sum_{N=1}^{NN} (v_{3,\alpha}^N + \overset{\circ}{b}{}_\alpha^\lambda v_\lambda^N)\overset{+}{v}^N = \sum_{N=1}^{NN} \varphi_{\alpha 3}^N \overset{+}{v}^N. \tag{26}$$

The shape functions of the dependent variables $\overset{+}{w}_i$ are defined by

$$\overset{+}{w}_\beta = \sum_{N=1}^{NN} w_\beta^N \overset{+}{v}^N, \quad \overset{+}{w}_3 = \sum_{N=1}^{NN} w_3^N \overset{+}{v}^N. \tag{27}$$

In order to find suitable equations for their calculation we substitute (26) and (27) into relations (19). In view of the arbitrariness of the quantities $\overset{+}{v}^N$, this leads to

$$w_\rho^N l_{\alpha\cdot}^{\rho} = -[\varphi_{\alpha\rho}^N w^\rho + \varphi_{\alpha 3}^N(1+w_3)], \quad w_3^N(1+w_3) = -w_\alpha^N w^\alpha. \tag{28}$$

These equations can be solved at each integral point for the values $N = 1, 2, \ldots, NN$ to give the corresponding values of w_i^N. By a similar transformation of the variational equations of second order (20), suitable relations can be obtained for the pointwise determination of the shape functions w_ρ^{MN}:

$$\overset{++}{w}_\rho = \sum_{M=1}^{NN}\sum_{N=1}^{NN} w_\rho^{MN} \overset{+}{v}^M \overset{+}{v}^N, \quad \overset{++}{w}_3 = \sum_{M=1}^{NN}\sum_{N=1}^{NN} w_3^{MN} \overset{+}{v}^M \overset{+}{v}^N. \tag{29}$$

Considering equations (26), (27) and (28), relations (17) and (18) can now be transformed into:

$$\overset{+}{\omega}_{\alpha\beta} = -\sum_{N=1}^{NN} [(1+w_3)(\varphi^N_{\alpha 3}|_\beta + \overset{\circ}{b}{}^\rho_\beta \varphi^N_{\alpha\rho}) + w^\rho(\varphi^N_{\alpha\rho}|_\beta - \overset{\circ}{b}_{\rho\beta}\varphi^N_{\alpha 3})$$
$$+ (\overset{\circ}{b}_{\alpha\beta} + \varphi_{\alpha 3}|_\beta + \overset{\circ}{b}{}^\rho_\beta \varphi_{\alpha\rho}) w^N_3 + (\varphi_{\alpha}{}^\rho|_\beta - \overset{\circ}{b}{}^\rho_\beta \varphi_{\alpha 3}) w^N_\rho] \overset{+}{v}{}^N$$
$$= \sum_{N=1}^{NN} \omega^N_{\alpha\beta} \overset{+}{v}{}^N,$$

$$\overset{++}{\omega}_{\alpha\beta} = -2 \sum_{M=1}^{NN}\sum_{N=1}^{NN} \left[w^M_3 (\varphi^N_{\alpha 3}|_\beta + \overset{\circ}{b}{}^\rho_\beta \varphi^N_{\alpha\rho}) + w^M_\rho (\varphi^{N\rho}_{\alpha}|_\beta - \overset{\circ}{b}{}^\rho_\beta \varphi^N_{\alpha 3}) \right.$$
$$\left. + (\overset{\circ}{b}_{\alpha\beta} + \varphi_{\alpha 3}|_\beta + \overset{\circ}{b}{}^\rho_\beta \varphi_{\alpha\rho}) \frac{w^{MN}_3}{2} + (\varphi_{\alpha}{}^\rho|_\beta - \overset{\circ}{b}{}^\rho_\beta \varphi_{\alpha 3}) \frac{w^{MN}_\rho}{2} \right] \overset{+}{v}{}^M \overset{+}{v}{}^N$$
$$= \sum_{M=1}^{NN}\sum_{N=1}^{NN} \omega^{MN}_{\alpha\beta} \overset{+}{v}{}^M \overset{+}{v}{}^N \tag{30}$$

where $\omega^N_{\alpha\beta}$ and $\omega^{MN}_{\alpha\beta}$ are the shape functions. All variables of the FS appearing in the above relations have to be calculated from the exact non-linear equations of the finite rotation theory.

The other shape functions can be developed using a similar procedure. Substituting the corresponding expressions into (16) we first obtain:

$$\sum_{M=1}^{NN}\sum_{N=1}^{NN} \delta\overset{+}{v}{}^N \Bigg\{ \underbrace{\iint_{\mathring{F}} (\tilde{N}^{(\alpha\beta)M} \alpha^N_{\alpha\beta} + M^{(\alpha\beta)M} \omega^N_{\alpha\beta})\, d\mathring{F}}_{k^{MN}_e}$$
$$\underbrace{+ \iint_{\mathring{F}} (\tilde{N}^{(\alpha\beta)} \alpha^{MN}_{\alpha\beta} + M^{(\alpha\beta)} \omega^{MN}_{\alpha\beta})\, d\mathring{F} + \oint_{\mathring{C}} \left[\frac{w^N_3}{(1+w_3)^2} (m_t \omega^M_t + m_u \omega^M_u) - \frac{1}{(1+w_3)} (m_t \omega^{MN}_t + m_u \omega^{MN}_u) \right] d\mathring{s}}_{k^{MN}_g} \Bigg\} \overset{+}{v}{}^M$$
$$= \sum_{N=1}^{NN} \delta\overset{+}{v}{}^N \Bigg\{ \underbrace{\iint_{\mathring{F}} (p^\alpha v^N_\alpha + p^3 v^N_3)\, d\mathring{F} + \oint_{\mathring{C}} \left[n_t v^N_t + n_u v^N_u + n_3 v^N_3 + \frac{1}{1+w_3} (m_t \omega^N_t + m_u \omega^N_u) \right] d\mathring{s}}_{p^N_e}$$
$$\underbrace{- \iint_{\mathring{F}} (\tilde{N}^{(\alpha\beta)} \alpha^N_{\alpha\beta} + M^{(\alpha\beta)} \omega^N_{\alpha\beta})\, d\mathring{F}}_{p^N_i} \Bigg\} \tag{31}$$

and hence

$$\sum_{M=1}^{NN} (k^{MN}_e + k^{MN}_g) \overset{+}{v}{}^M = p^N_e - p^N_i \rightarrow \sum_{M=1}^{NN} k^{MN}_t \overset{+}{v}{}^M = \Delta p^N, \quad N = 1, \ldots, NN \tag{32}$$

which represents the well-known stiffness relation of the geometrically non-linear finite element in terms of the following submatrices:

k_e^{MN} = elastic stiffness matrix,
k_g^{MN} = geometric stiffness matrix,
p_e^N = vector of external nodal forces,
p_i^N = vector of internal nodal forces,
k_t^{MN} = tangent stiffness matrix,
Δp^N = vector of unbalanced forces.

Due to the index-oriented formulation, the matrices given in eqn (31) have a very transparent form and show exactly the operations to be performed for their derivation. This is a significant advantage, in particular, for the numerical implementation.

The assembly of the so developed element matrices into the global stiffness matrix of the complete shell structure can be performed in a standard manner. After evaluation of the displacements $\overset{+}{v}_i$ from (32), the displacements $\bar{v}_i = v_i + \overset{+}{v}_i$ defining the FS of the subsequent iteration step have to be constructed in order to calculate the corresponding internal kinematic and force variables form the exact non-linear equations. It should, however, be mentioned that the internal forces occurring in the constitutive eqns (22) are pseudo-variables which cannot be directly interpreted on the element level. For practical applications therefore, they have to be transformed into physical components $N^{\langle\alpha\beta\rangle}$ and $M^{\langle\alpha\beta\rangle}$ which are true force variables measured per unit length of the deformed coordinate lines θ^α = const. The corresponding transformations are given by Başar (1986, 1987).

In accordance with the concept adopted for the finite element derivation, the integrations occurring in (31) have to be performed numerically. Replacing the surface and line elements $d\mathring{F}$ and $d\mathring{s}$ by the well-known expressions

$$d\mathring{F} = \sqrt{\mathring{a}}\, d\theta^1\, d\theta^2, \quad d\mathring{s} = \sqrt{\mathring{a}_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta} \tag{33}$$

the elastic stiffness matrix defined in (31) takes, for instance, the form:

$$\begin{aligned} k_e^{MN} &= \iint_{\mathring{F}} (\tilde{N}^{(\alpha\beta)M}\alpha_{\alpha\beta}^N + M^{(\alpha\beta)M}\omega_{\alpha\beta}^N)\sqrt{\mathring{a}}\, d\theta^1\, d\theta^2 \\ &= \sum_{I=1}^{II} ({}^I\tilde{N}^{(\alpha\beta)M}\,{}^I\alpha_{\alpha\beta}^N + {}^IM^{(\alpha\beta)M}\,{}^I\omega_{\alpha\beta}^N)\sqrt{{}^I\mathring{a}}\,{}^IWA', \end{aligned} \tag{34}$$

where

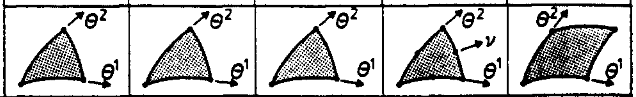
I = actual point of integration,
II = number of integration points per element,
${}^I(\ )$ = value of a variable at I,
${}^IW$ = weighting coefficient at I,
A' = area of the finite element with straight edges in the plane θ^α.

From (34) we see that the exact values of the geometrical variables $\mathring{a}_{\alpha\beta}$, $\mathring{b}_{\alpha\beta}, \ldots$ can be considered at every integral point I. This ensures a very accurate representation of the shell geometry and—due to a similar treatment of the factor $\sqrt{\mathring{a}}$—of the area $\mathring{F}$ of the curved finite element as well. The present tensor-oriented development of finite elements permits one to use the original non-linear shell theory in connection with various shape functions (23) for the displacements $\overset{+}{v}_\alpha$ and $\overset{+}{v}_3$. For the derivation of the triangular and rectangular curvilinear finite elements, of *the finite-rotation element family*, the HERMITE-polynomials shown in Table 1 have been used. A detailed discussion of their convergence behaviour is given by Harte (1982) and Harte and Eckstein (1986).

5. NUMERICAL RESULTS

A large number of examples has been analysed by the finite-rotation elements presented above. In the following, three examples are given to demonstrate their applicability to strongly non-linear situations.

Table 1. Family of finite-rotation elements

	FINITE-ROTATION ELEMENT				
	27	36	54	63	48
					
Polynomials v_λ v_3	ZIENKIEWICZ (see BAZELEY et al. (1965))	ZIENKIEWICZ COWPER	COWPER (1970)	BELL (1969)	bicubic HERMITE-polynomials
Degrees of freedom	v_i $v_{i,\alpha}$	v_i $v_{i,\alpha}$ $v_{3,\alpha\beta}$	v_i $v_{i,\alpha}$ $v_{i,\alpha\beta}$	v_i $v_{i,\alpha}$ $v_{i,\alpha\beta}$ $v_{i,\nu}$	v_i $v_{i,\alpha}$ $v_{i,12}$
	3x9	3x12	3x18	3x18 3x3	4x12
Points of integration	12	21	21	21	16

The triangular cantilever plate under a concentrated load couple (Fig. 2) has been analysed up to a load level for which the free end B undergoes a rotation of 540°. Comparison of the numerical results with the available analytical solution shows the remarkable accuracy of the finite-rotation elements used (Fig. 3). Even when the behavior becomes exceedingly non-linear the numerical errors are almost negligible (< 1%). The analysis of this example and the following ones has been performed by the finite-rotation element 54 (Table 1).

The factor $1/(1+w_3)$ occurring in the incremental principle (16) tends to infinity for $\omega \to +\pi/2$. This causes, however, no numerical difficulties in the analysis. Even for the load level corresponding to a theoretical rotation angle $\omega = 90°$, the computation had led to a very accurate value $\omega = 89.97°$.

The spherical shell (Fig. 4) under a concentrated load has been used by Nolte (1983) for a systematic comparison of finite elements based on various non-linear shell theories of

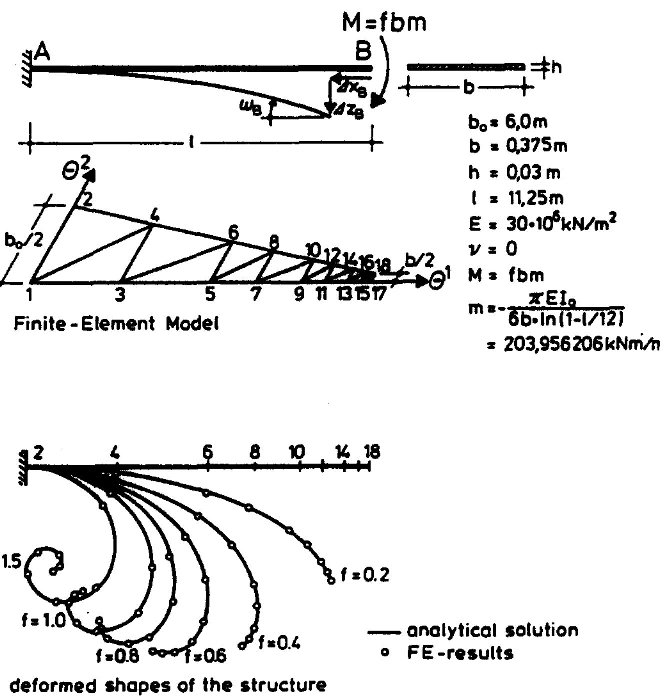


Fig. 2. Triangular cantilever plate subjected to a load couple.

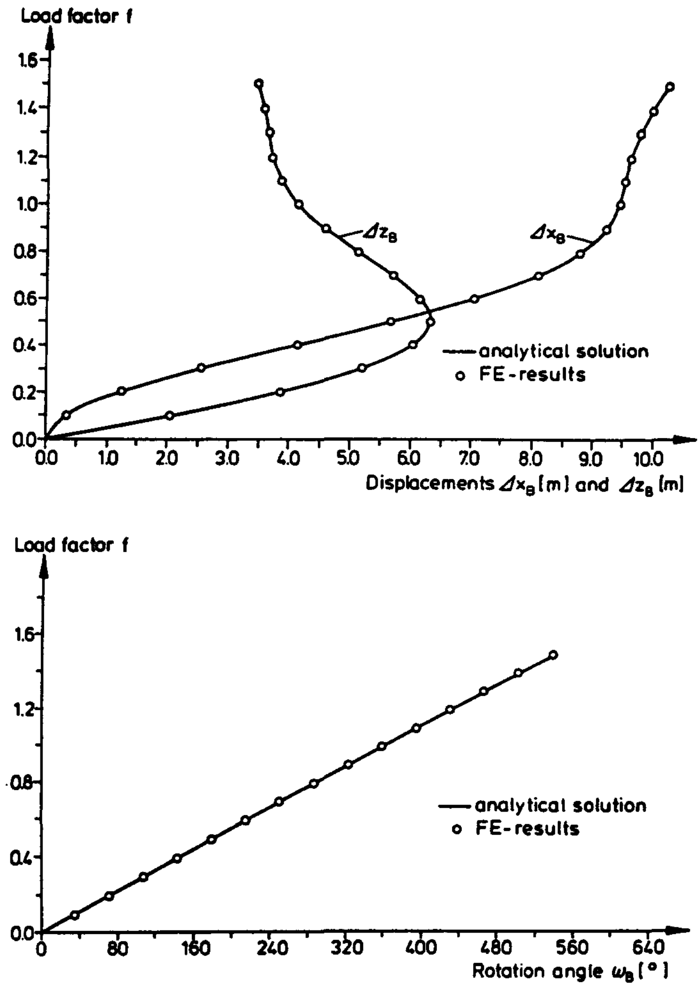


Fig. 3. Numerical results of triangular cantilever plate subjected to a load couple.

different orders of accuracy. In the whole non-linear domain, our results are in full agreement with those obtained by Nolte (1983) on the basis of his own finite-rotation theory. This can be seen as a confirmation of the accuracy of both formulations. Another remarkable result is the fact that the DONNELL–MARGUERRE theory, which represents the simplest approximate non-linear shell theory, may give better results than the moderate-rotation theory given by Başar and Krätzig (1985) and Harte (1982). It seems that in this example the assumptions involved in the DONNELL–MARGUERRE theory are better satisfied than those adopted for the derivation of the moderate-rotation theory in question. This shows the importance of using finite-rotation elements rather than moderate-rotation ones when dealing with arbitrary non-linear phenomena.

Figure 5 shows a hyperbolic paraboloidal shell supported at the two single points A and B in such a way that it can undergo displacements in the corresponding normal directions $\mathring{a}_3$. The loading consists of two opposite point moments which upon increasing will cause the two supports to approach each other. This leads to a considerable vertical displacement as well as to considerable bending stresses in the neighborhoods of the supports. For $f = 7.0$ the rotation of the support points amounts to an angle of $\omega = 620°$ while the vertical displacement of the upper point D reaches the value $v_{\langle 3 \rangle} = 18.57$ m which is comparable to the span AB = 27.40 m (Figs 6, 7). This highly non-linear problem is also characterized by quasi-inextensional bending which is known to be numerically very sensitive. The triangular finite element 54 (Table 1) used here was capable of dealing with this problem without any numerical difficulties. Figure 6 shows the deformed configurations of section $\theta^2 = 0$ for different values of f. In Fig. 7 numerical results are presented for

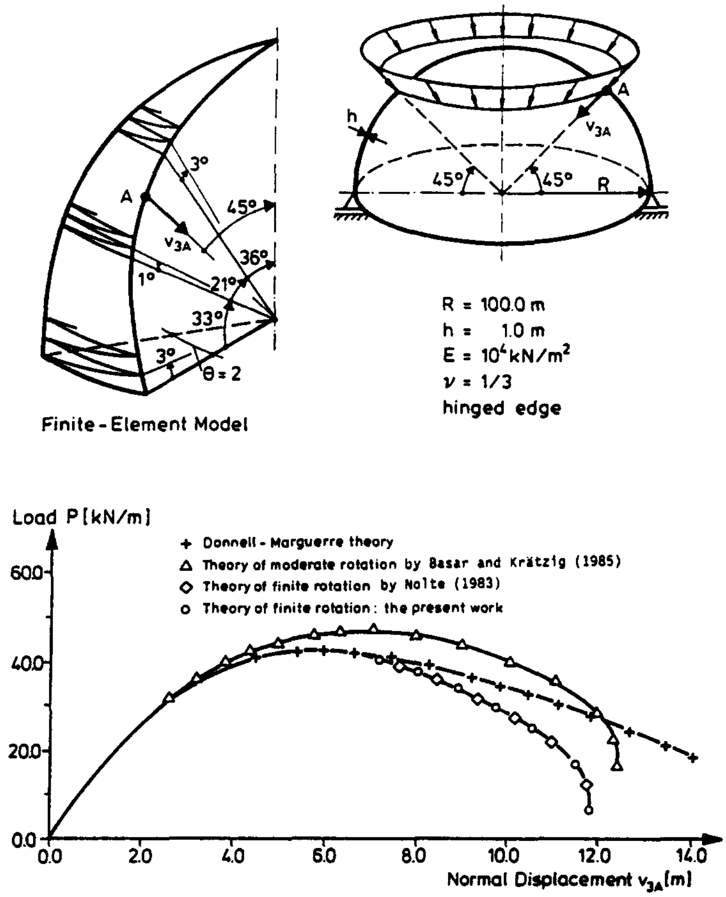


Fig. 4. Spherical shell under ring load.

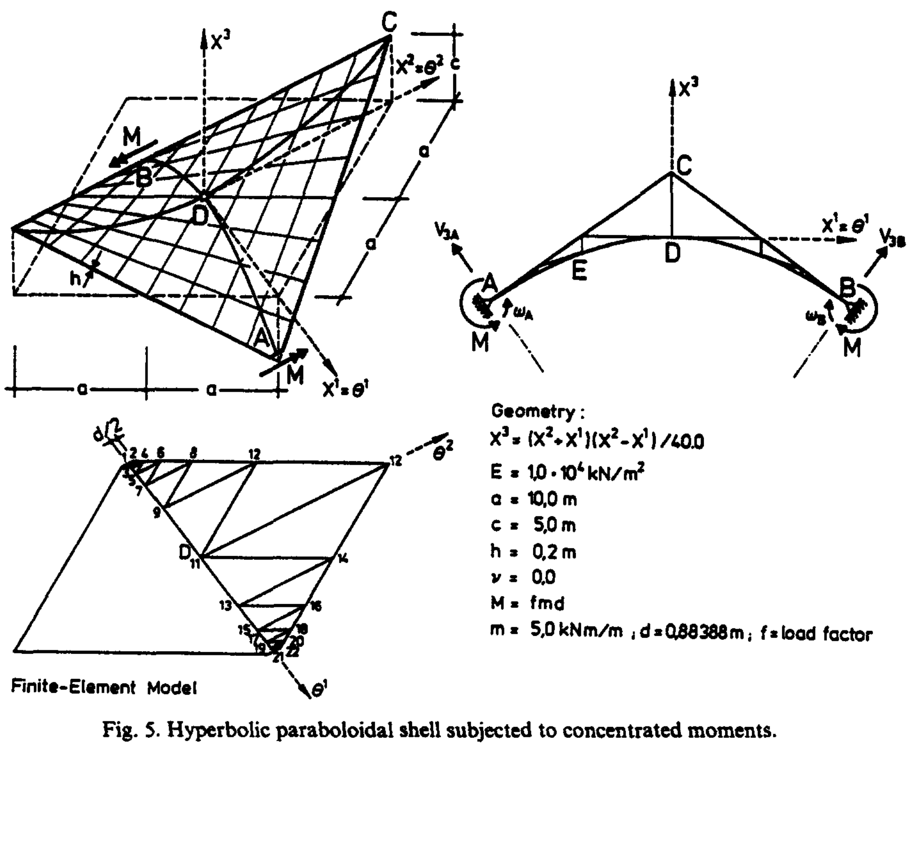


Fig. 5. Hyperbolic paraboloidal shell subjected to concentrated moments.

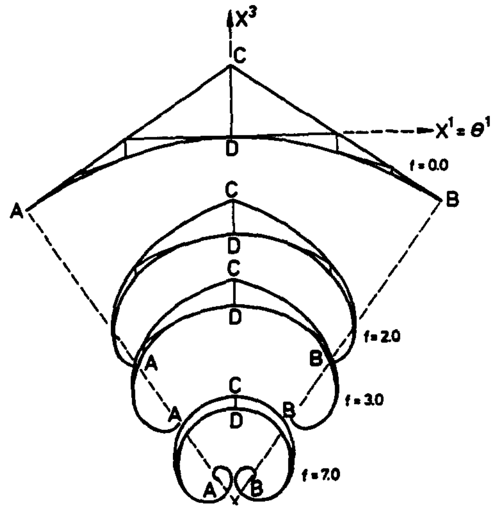


Fig. 6. Deformed shapes of the normal section $\theta^2 = 0$ of a hyperbolic paraboloidal shell.

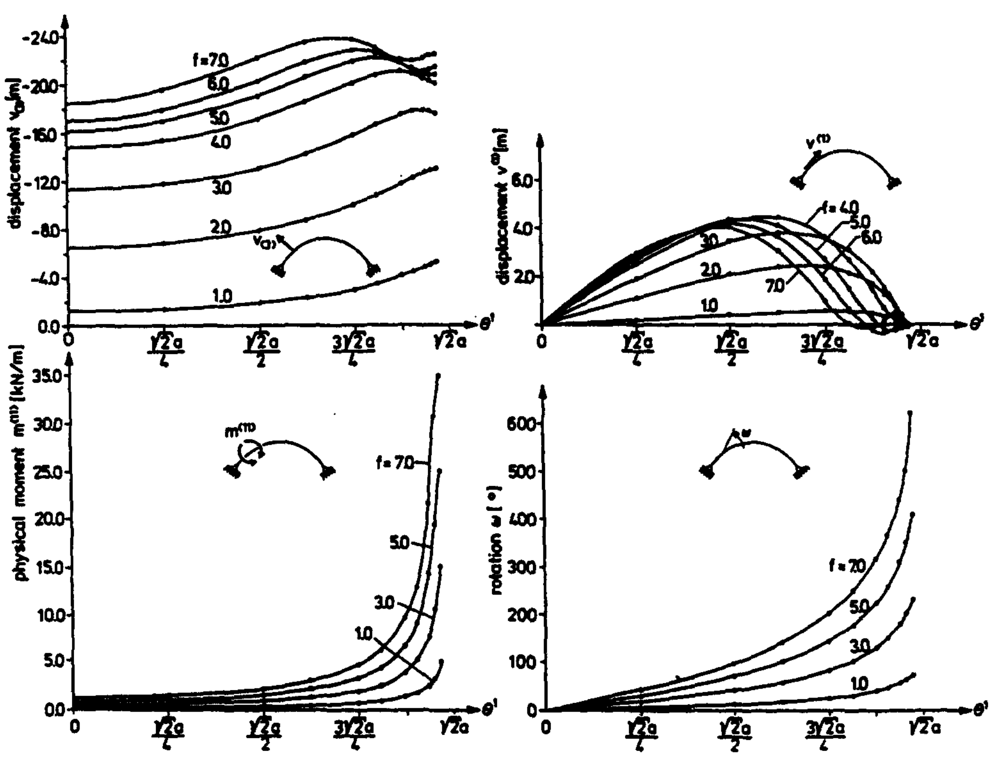


Fig. 7. Numerical results of a hyperbolic paraboloidal shell.

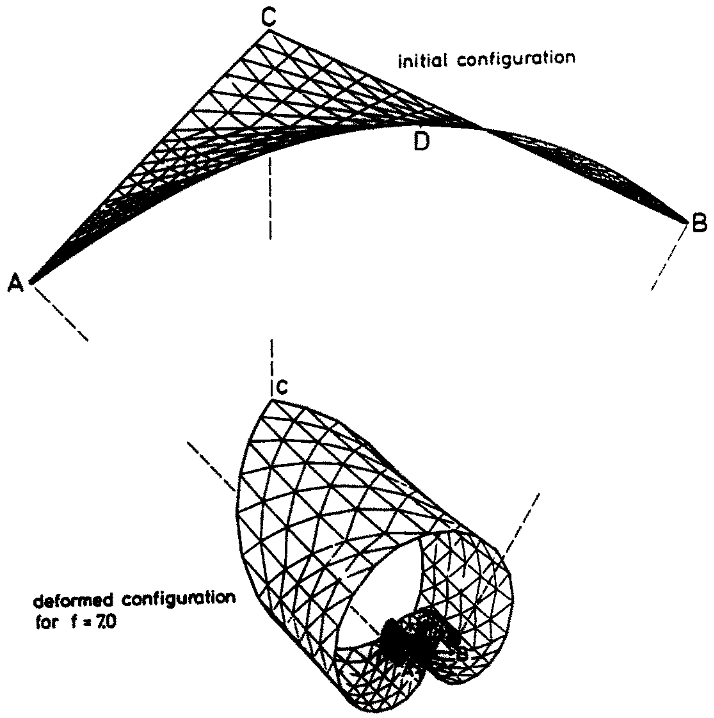


Fig. 8. Undeformed and deformed configuration of a hyperbolic paraboloidal shell for $f = 7.0$.

several characteristic variables, again along the coordinate line $\theta^2 = 0$. Finally, a three-dimensional plot of the deformed configuration of the whole shell for $f = 7.0$ is given in Fig. 8.

6. CONCLUSIONS

Starting from a consistent, finite-rotation shell theory by Başar (1987) a family of finite-rotation shell elements has been derived. The element formulation which is based on a tensor-oriented procedure appears to have the following advantages.

(1) It enables the use of various interpolation functions for the displacement field and thus a unified derivation of an entire element family.
(2) To derive the element matrices only tensor operations must be performed, once the shape functions of the displacements and their derivatives have been determined.
(3) Arbitrary shell geometries can easily be considered,

Extended numerical studies indicate that the finite-rotation elements presented here may be qualified as follows.

(4) Remarkable accuracy may be achieved, even in the treatment of strong non-linearities. In one instance the rotational angle even reached 620°.
(5) In the analysis of pure bending problems membrane locking did not occur.
(6) Due to the accuracy of the underlying theoretical formulation, non-linear phenomena induced by load couples can be analysed for arbitrary load values. In the case of moderate-rotation elements by Harte and Eckstein (1986) this is not possible.

As regards the mechanical significance of the results we may state the following.

(7) The numerical results obtained for a spherical cap (Fig. 4) are identical with those of Nolte (1983) and may thus be regarded as a confirmation of the quality of the theoretical formulations presented by Nolte (1983) and Başar (1987). Such agreements are rather exceptional, especially in the strongly non-linear range.

(8) When short-wave deformation patterns are predominant, moderate-rotation finite elements may give less accurate results than those derived on the basis of the DONNELL–MARGUERRE theory, which is the simpliest non-linear shell theory

This last result shows clearly that a reliable analysis of non-linear phenomena should be performed using finite-rotation elements.

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